On virtual link invariants

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Abstract. We characterize the virtual link invariants that are partition functions of vertex models (as considered by de la Harpe and Jones), both in the real and in the complex case. We show that for any fixed number of states, these invariants form an affine variety. Basic techniques are the first and second fundamental theorem of invariant theory for the orthogonal group (in the sense of Weyl) and some related methods from algebraic geometry.

1. Introduction and survey of results

This paper is inspired by some recent results in the range of combinatorial parameters and invariant theory, in particular results of Szegedy [12], Freedman, Lovász, and Schrijver [2], and Schrijver [10,11]. The concepts of virtual link and virtual link diagram were introduced by Kauffman [6]; see Kauffman [7] for more background, which we partially repeat below.

A *virtual link diagram* is an undirected graph G such that for each vertex v:

(1) v has degree 4 and the edges incident with v are cyclically ordered, where one pair of opposite edges is labeled as 'over-going'.

G may have loops and multiple edges. Moreover, 'vertexless' loops are allowed, that is, loops without a vertex. We denote the vertexless loop by O. Let $\mathcal{G}$ denote the collection of virtual link diagrams, two of them being the same if they are isomorphic.

A virtual link diagram can be seen as the projection of a link in $M \times \mathbb{R}$ where M is some oriented surface. Since this connection however is not stable under all Reidemeister moves (e.g., one may need to create a handle to allow a type II Reidemeister move), we will view virtual link diagrams just abstractly as given above.

In this paper, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and

(2) $$[n] := \{1, \ldots, n\}$$

for any $n \in \mathbb{N}$. We denote the sets of vertices and edges of a virtual link diagram G by VG and EG, respectively. K_0 denotes the virtual link diagram with no vertices and edges.

Let V be an n-dimensional complex linear space with a symmetric nondegenerate bilinear form $\langle .,. \rangle$. We can identify V with $\mathbb{C}^n$, with the standard bilinear form $\langle x, y \rangle = x^{\mathsf{T}} y$. Having the bilinear form, we can identify V^* with V.

Let S_2 act on $V^{\otimes 4}$ so that the nonidentity element of S_2 brings $x_1 \otimes x_2 \otimes x_3 \otimes x_4$ to $x_3 \otimes x_4 \otimes x_1 \otimes x_2$. Define

(3) $$\mathcal{M}_n := (V^{\otimes 4})^{S_2},$$

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that is, the linear space of S_2-invariant elements of $V^{\otimes 4}$. Note that $\mathcal{M}_n$ can be identified with the collection of symmetric matrices in $(\mathbb{C}^{n\times n})^{\otimes 2}$.

Following de la Harpe and Jones [4], any element R of $\mathcal{M}_n$ can be called a *vertex model.* For any $R \in \mathcal{M}_n$, let f_R be the *partition function* of R; that is, the virtual link diagram invariant defined by

$$f_R(G) = \sum_{\phi: EG\to[n]} \prod_{v\in VG} R_{\phi(\delta(v))}. \tag{4}$$

Here we put

$$\phi(\delta(v)) := (\phi(e_1), \phi(e_2), \phi(e_3), \phi(e_4)), \tag{5}$$

where e_1, e_2, e_3, e_4 are the edges incident with v, in clockwise order, and where e_1, e_3 is the over-going pair. Since R is S_2-invariant, $R_{\phi(\delta(v))}$ is well-defined.

We call $f : \mathcal{G} \to \mathbb{C}$ an *n-state partition function* if $f = f_R$ for some $R \in \mathcal{M}_n$. In Corollary 1a we characterize n-state partition functions.

2. Reidemeister moves

Reidemeister moves yield an isotopy of virtual link diagrams. A *virtual link* is an isotopy class of virtual link diagrams. A *virtual link diagram invariant* is a function defined on virtual link diagrams that is invariant under isomorphisms. A *virtual link invariant* is a virtual link diagram invariant that is invariant under Reidemeister moves. (So in fact it is a function on virtual links, but the definition as given turns out to be more convenient.)

We recall the well-known sufficient conditions for f_R to be a virtual link invariant (i.e., to be invariant under Reidemeister moves). To this end, let $C : (\mathbb{C}^{n\times n})^{\otimes 2} \to \mathbb{C}^{n\times n}$, $D : (\mathbb{C}^{n\times n})^{\otimes 2} \to (\mathbb{C}^{n\times n})^{\otimes 2}$, and $E_{1,2}, E_{1,3}, E_{2,3} : (\mathbb{C}^{n\times n})^{\otimes 2} \to (\mathbb{C}^{n\times n})^{\otimes 3}$ be the unique linear functions satisfying

$$\begin{aligned} &C(M\otimes N) = MN,\ D(M\otimes N) = M\otimes N^{\mathsf{T}},\ E_{1,2}(M\otimes N) = M\otimes N\otimes I_n,\\ &E_{1,3}(M\otimes N) = M\otimes I_n\otimes N,\ E_{2,3}(M\otimes N) = I_n\otimes M\otimes N \end{aligned} \tag{6}$$

for all $M, N \in \mathbb{C}^{n\times n}$, where I_n denotes the identity matrix in $\mathbb{C}^{n\times n}$. Then a *sufficient* condition for f_R to be a virtual link invariant is (cf. Turaev [13], Kauffman [5]):

$$C(R) = I_n,\ RD(R) = I_n\otimes I_n,\ E_{1,2}(R)E_{1,3}(R)E_{2,3}(R) = E_{2,3}(R)E_{1,3}(R)E_{1,2}(R). \tag{7}$$

The last equation in (7) is the *Yang-Baxter equation.*

In the real case, condition (7) is necessary and sufficient for f_R being invariant under Reidemeister moves. But in the complex case, condition (7) is only a sufficient, and not a necessary condition for f_R being invariant under Reidemeister moves. For instance, if $R = A\otimes A$ with $A := \begin{pmatrix} 2 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ (where i is the imaginary unit), then $f_R(G)$ is equal to 2^k where k is the number of knots in the virtual link G. (A *knot* in a virtual link diagram

(V,E) is a component of the graph on E where two edges are adjacent if and only if there is a vertex where they are oppositie.) So f_R is invariant under Reidemeister moves. Yet, $C(R) \neq I_2$.

We will however see that there exists for each R with f_R invariant under Reidemeister moves an R' with $f_{R'} = f_R$ and R' satisfying (7). This holds in fact more generally for any linear combination of 'tangles' that leaves f_R invariant.

3. Some notation and terminology

Let $\mathbb{C}\mathcal{G}$ denote the space of formal linear combinations of elements of $\mathcal{G}$. The elements of $\mathbb{C}\mathcal{G}$ are called *quantum virtual link diagrams*. Any function on $\mathcal{G}$ with values in a $\mathbb{C}$-linear space can be extended linearly to $\mathbb{C}\mathcal{G}$. Let GH denote the disjoint union of G and H. Then, taking GH as products, makes $\mathbb{C}\mathcal{G}$ to an algebra.

As usual, let $\mathcal{O}(X)$ denote the set of all regular functions on a linear space X. Define $p_n : \mathcal{G} \to \mathcal{O}(\mathcal{M}_n)$ by

$$(8) \qquad p_n(G)(X) := f_X(G)$$

for $G \in \mathcal{G}$ and $X \in \mathcal{M}_n$. Note that $p_n(K_0) = 1$ and

$$(9) \qquad p_n(G)p_n(H) = p_n(GH).$$

Extending p_n to $\mathbb{C}\mathcal{G}$, makes p_n to an algebra homomorphisms $\mathbb{C}\mathcal{G} \to \mathcal{O}(\mathcal{M}_n)$.

A central step in our proof is describing the image Im p_n and the kernel Ker p_n of p_n. This is based on the First and Second Fundamental Theorem of Invariant Theory (in the sense of Weyl [14]) for the orthogonal group (the 'FFT' and 'SFT') — cf. Goodman and Wallach [3].

4. Tangles

Let $k \in \mathbb{N}$. A *k-tangle* is an undirected graph where each vertex v either satisfies (1) or has degree 1, and where there are precisely k vertices of degree 1, labeled $1, \ldots, k$. (It follows that k is even.) Let $\mathcal{T}_k$ and $\mathcal{T}$ denote the collections of k-tangles and of tangles, respectively. A *tangle* is a k-tangle for any k. The 0-tangles are precisely the virtual link diagrams, and so $\mathcal{T}_0 = \mathcal{G}$.

A *quantum k-tangle* is a formal $\mathbb{C}$-linear combination of k-tangles, and a *quantum tangle* a formal linear combination of tangles. Let $\mathbb{C}\mathcal{T}_k$ and $\mathbb{C}\mathcal{T}$ denote the spaces of quantum k-tangles and quantum tangles, respectively. (So $\mathbb{C}\mathcal{T}_0 = \mathbb{C}\mathcal{G}$.) Any function on the set of tangles with values in a $\mathbb{C}$-linear space can be linearly extended to $\mathbb{C}\mathcal{T}$.

Let u and v be vertices of degree 1 in a tangle. We say that we *glue* u and v if we identify u and v and ignore them as vertices, joining the incident edges to one edge. If T and T' are k-tangles, then $T \cdot T'$ is defined to be the 0-tangle obtained from the disjoint union of T and T' by glueing the vertices labeled i in T and T', for $i = 1, \ldots, k$. By defining $T \cdot T' = 0$ for any k-tangle T and l-tangle T' with $k \neq l$, we obtain a bilinear form $\mathbb{C}\mathcal{T} \times \mathbb{C}\mathcal{T} \to \mathbb{C}\mathcal{G}$.

5. Applying invariant theory

As usually, for any group G acting on a set S, let $S^G := \{x \in S \mid x^U = x \text{ for each } U \in G\}$.

Let O_n denote the group of complex orthogonal matrices. O_n acts on $V^{\otimes k}$ by $x^U = U^{\otimes k} x$ for $x \in V^{\otimes k}$. In particular, O_n acts on $\mathcal{M}_n$. This induces an action on $\mathcal{O}(\mathcal{M}_n)$. Note that

$$f_{R^U}(G) = f_R(G) \tag{10}$$

for each virtual link diagram G and each $U \in O_n$. Hence $p_n(G)$ is an O_n-invariant polynomial for each G. As we will see, these span all O_n-invariant polynomials in $\mathcal{O}(\mathcal{M}_n)$.

For each $\pi \in S_n$, let T_π be the $2n$-tangle with vertex set $[2n]$, labeled by $\lambda : [2n] \to [2n]$ with $\lambda(i) = i$ for $i \in [2n]$, and edges $\{i, n + \pi(i)\}$ for each $i \in [n]$. Define the quantum $2n$-tangle $\det_n$ by

$$\det{}_n := \sum_{\pi \in S_n} \operatorname{sgn}(\pi) T_\pi. \tag{11}$$

Define

$$J_n := \det{}_n \cdot \mathbb{C}\mathcal{T}_{2n}. \tag{12}$$

Theorem 1. Im $p_n = \mathcal{O}(\mathcal{M}_n)^{O_n}$ *and* Ker $p_n = J_{n+1}$.

Proof. It is easy to see that each $p_n(G)$ is O_n-invariant. This gives $\subseteq$ in the first equality. One also checks directly $\supseteq$ in the second equality.

To see the reverse inclusions, choose $k \in \mathbb{N}$, and let $\mathcal{G}_k$ be the collection of virtual link diagram with k vertices. Let $\mathcal{O}_k(\mathcal{M}_n)$ be the set of all homogeneous polynomials in $\mathcal{O}(\mathcal{M}_n)$ of degree k.

Let $m := 4k$. Let $S\mathbb{C}^{m \times m}$ be the set of symmetric matrices in $\mathbb{C}^{m \times m}$. Let $\mathcal{L}$ be the subspace of $\mathcal{O}(S\mathbb{C}^{m \times m})$ spanned by the monomials $\prod_{ij \in M} x_{ij}$ where M is any perfect matching on $[m]$. We give each $j \in [m]$ label j. This makes any perfect matching M on $[m]$ to an m-tangle.

We define linear functions τ, μ, and σ so as to make a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}\mathcal{G}_k & \xrightarrow{p} & \mathcal{O}_k(\mathcal{M}_n) \\ \uparrow \mu & & \uparrow \sigma \\ \mathcal{L} & \xrightarrow{\tau} & (\mathbb{C}^n)^{\otimes m} \end{array} \tag{13}$$

First, define μ and τ by

$$\mu\Big(\prod_{ij \in M} x_{ij}\Big) := T \cdot M \qquad \text{and} \qquad \tau\Big(\prod_{ij \in M} x_{ij}\Big) := \sum_{\substack{\phi : [m] \to [n] \\ \forall e \in M : |\phi(e)| = 1}} e_\phi, \tag{14}$$

for any perfect matching M on $[m]$. Here T is the m-tangle with labeled vertices $1,\ldots,m$ and unlabeled vertices $v_1,\ldots,v_k$, with v_i connected to vertices $4i-3, 3i-2, 4i-1, 4i$, in this order. Moreover,

$$e_\phi := \bigotimes_{h\in[m]} e_{\phi(h)}. \tag{15}$$

Next, define σ by

$$\sigma(e_\phi) := \prod_{i\in[k]} \rho_{S_2}(e_{\phi_i}) \tag{16}$$

for $\phi : [m] \to [n]$. Here ρ_{S_2} is the Reynolds operator for S_2, and ϕ_i is the function $[4] \to [n]$ with $\phi_i(j) := \phi(4i-4+j)$ for $j \in [4]$.

Now diagram (13) commutes, that is,

$$p \circ \mu = \sigma \circ \tau. \tag{17}$$

To prove it, choose a perfect matching M on $[m]$. Then $p(\mu(\prod_{ij\in M} x_{ij})) = p(T \cdot M)$. Moreover,

$$\sigma(\tau(\prod_{ij\in M} x_{ij})) = \sum_{\substack{\phi:[m]\to[n] \\ \forall e\in M:|\phi(e)|=1}} \sigma(e_\phi) = \sum_{\substack{\phi:[m]\to[n] \\ \forall e\in M:|\phi(e)|=1}} \prod_{i\in[k]} \rho_{S_2}(e_{\phi_i}) = p(T \cdot M). \tag{18}$$

This proves (17).

Now by the FFT, Im $\tau = ((\mathbb{C}^n)^{\otimes m})^{O_n}$. Moreover, Im $\mu = \mathbb{C}\mathcal{G}_k$ and Im $\sigma = \mathcal{O}_k(\mathcal{M}_n)$. Hence

$$p(\mathbb{C}\mathcal{G}_k) = \text{Im } (p\circ\mu) = \text{Im } (\sigma\circ\tau) = \sigma(\text{Im } \tau) = \sigma((\mathbb{C}^n)^{\otimes m})^{O_n} = (\mathcal{O}_k(\mathcal{M}_n))^{O_n}. \tag{19}$$

So Im $p = \mathcal{O}(\mathcal{M}_n)^{O_n}$.

By the SFT, Ker τ is equal to the ideal I in $\mathcal{O}(S\mathbb{C}^{m\times m})$ generated by the $(n+1)\times(n+1)$ minors of $S\mathbb{C}^{m\times m}$. Choose $\gamma \in \mathbb{C}\mathcal{G}_k$ with $p(\gamma) = 0$. Then $\gamma = \mu(q)$ for some $q \in \mathcal{L}^{\mathcal{S}}$, where $\mathcal{S}$ is the group of permutations of $[m]$ generated by S_k and by S_2. Hence $\tau(q) \in ((\mathbb{C}^n)^{\otimes m})^{\mathcal{S}}$. As $\sigma(\tau(q)) = p(\mu(q)) = p(\gamma) = 0$, this implies $\tau(q) = 0$. Hence $q \in I$. Therefore, $\gamma = \mu(q) \in \mu(I) \subseteq J_{n+1}$. ∎

Corollary 1a. *Let f be a virtual link diagram invariant and let $n \in \mathbb{N}$. Then f is an n-state partition function if and only if f is multiplicative and $f(\det_{n+1} \cdot T) = 0$ for each $2(n+1)$-tangle T.*

Proof. Necessity follows from the fact that if $f = f_R$, then $f(G) = p_n(G)(R)$ for each G. So f is multiplicative as p_n is an algebra homomorphism. Moreover, for any $2(n+1)$-tangle T one has $f_R(\det_n \cdot T) = p_n(\det_n \cdot T)(R) = 0$, as $\det_n \cdot T$ belongs to Ker p_n.

To see sufficiency, as $f(\text{Ker } p_n) = 0$, there exists an algebra homomorphism $\hat{f} : \mathcal{O}(\mathcal{M}_n)^{O_n} \to \mathbb{C}$ such that $\hat{f} \circ p_n = f$. Now $1 \notin \text{Ker } \hat{f}$. Hence, since O_n is reductive, Ker $\hat{f}$ has a common zero R. Then for each virtual link diagram G, $f_R(G) - f(G) = (p_n(G) - f(G))(R) = 0$, as $p_n(G) - f(G)$ belongs to Ker $\hat{f}$. ∎

6. Extension to tangles

Let $R \in \mathcal{M}_n$ and let T be a k-tangle. Define the tensor $f_R(T)$ in $V^{\otimes k}$ by

$$f_R(T) := \sum_{\kappa : ET \to [n]} \left(\prod_{v \in V'T} R_{\kappa(\delta(v))} \right) e_{\kappa(\varepsilon_1)} \otimes \cdots \otimes e_{\kappa(\varepsilon_k)}. \tag{20}$$

Here ε_i denotes the edge incident with the vertex labeled i, for $i = 1, \ldots, k$. Moreover,

$$V'T := \text{set of unlabeled vertices of } T. \tag{21}$$

Define

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}. \tag{22}$$

The set $\mathcal{O}(\mathcal{M}_n) \otimes V^{\otimes k}$ is naturally equivalent to the set of all morphisms $\mathcal{M}_n \to V^{\otimes k}$. (The element $p \otimes v$ with $p \in \mathcal{O}(\mathcal{M}_n)$ and $v \in V^{\otimes k}$ corresponds to the function $x \mapsto p(x)v$.) Then O_n-invariant elements in $\mathcal{O}(\mathcal{M}_n) \otimes T(V)$ correspond to the O_n-equivariant morphisms $\mathcal{M}_n \to T(V)$.

The bilinear form on V induces a bilinear form $\cdot : T(V)^2 \to \mathbb{C}$, which induces a bilinear form $\cdot : (\mathcal{O}(\mathcal{M}_n) \otimes T(V))^2 \to \mathcal{O}(\mathcal{M}_n)$. It brings pairs of O_n-equivariant morphisms to O_n-invariant polynomials in $\mathcal{O}(\mathcal{M}_n)$.

For any k-tangle T, define the element $p_n(T)$ of $\mathcal{O}(\mathcal{M}_n) \otimes V^{\otimes k}$ by

$$p_n(T)(X) := f_X(T) \tag{23}$$

for $X \in \mathcal{M}_n$. Note that this extends the definitions of $p_n(G)$ and $f_R(G)$ given above. Then for quantum tangles τ, τ':

$$p_n(\tau \cdot \tau') = p_n(\tau) \cdot p_n(\tau'). \tag{24}$$

Theorem 2. $p_n(\mathbb{C}\mathcal{T}) = (\mathcal{O}(\mathcal{M}_n) \otimes T(V))^{O_n}$.

Proof. This can be proved similarly to Theorem 1, now taking $m = 4k + l$. ∎

7. Nondegeneracy

Define

$$\mathcal{W}_n := \{R \in \mathcal{M}_n \mid \text{the bilinear form} \cdot \text{on } f_R(\mathbb{C}\mathcal{T}) \text{ is nondegenerate}\}. \tag{25}$$

The condition is equivalent to saying that for each quantum tangle τ: if $f_R(\tau \cdot T') = 0$ for each tangle T', then $f_R(\tau) = 0$.

Theorem 3. *For each $R \in \mathcal{M}_n$ there exists $R' \in \mathcal{W}_n$ with $f_{R'}(G) = f_R(G)$ for each virtual link diagram G.*

Proof. Let $\mathcal{C}$ be the collection of all quantum tangles τ such that $f_R(\tau \cdot T') = 0$ for each tangle T'. We first show that the $p_n(\tau)$ (over all $\tau \in \mathcal{C}$) have a common zero.

Suppose such a common zero does not exist. Then by the Nullstellensatz there exists a finite subset $\mathcal{C}_0$ of $\mathcal{C}$ and for each $\tau \in \mathcal{C}_0$ a $q_\tau \in \mathcal{O}(\mathcal{M}_n) \otimes T(V)$ with $\sum_{\tau \in \mathcal{C}_0} p_n(\tau) \cdot q_\tau = 1$. Applying the Reynolds operator we can assume that each q_τ belongs to $(\mathcal{O}(\mathcal{M}_n) \otimes T(V))^{O_n}$. So by Theorem 2, $q_\tau = p_n(\tau')$ for some tangle τ' (depending on τ). Then

$$1 = \sum_{\tau \in \mathcal{C}_0} p_n(\tau) \cdot p_n(\tau') = \sum_{\tau \in \mathcal{C}_0} p_n(\tau \cdot \tau'). \tag{26}$$

However, $p_n(\tau \cdot \tau')(R) = 0$ for each $\tau \in \mathcal{C}$, since $f_R(\tau \cdot \tau') = 0$. So (26) does not hold.

So the $p_n(\tau)$ have a common zero R'. Then $f_{R'}(G) = f_R(G)$ for any virtual link diagram G, since $G - f_R(G)K_0$ belongs to $\mathcal{C}$, hence

$$0 = f_{R'}(G - f_R(G)K_0) = f_{R'}(G) - f_R(G)f_{R'}(K_0) = f_{R'}(G) - f_R(G). \tag{27}$$ ∎

8. Uniqueness

Theorem 1 implies that (for each fixed n) the ring $p_n(\mathbb{C}\mathcal{G}) \cong \mathbb{C}\mathcal{G}/J_{n+1}$ is finitely generated, and that the set $\{f_R \mid R \in \mathcal{M}_n\}$ of all n-state partition functions $f : \mathcal{G} \to \mathbb{C}$ form the affine variety $\mathcal{M}_n/O_n$, since $f_R = f_{R'}$ if and only if $p(R) = p(R')$ for each $p \in \mathcal{O}(\mathcal{M}_n)^{O_n}$. So

$$\mathcal{O}(\mathcal{M}_n/O_n) = \mathcal{O}(\mathcal{M}_n)^{O_n} \cong \mathbb{C}\mathcal{G}/J_{n+1}, \tag{28}$$

and the elements of $\mathcal{M}_n/O_n$ are in one-to-one correspondence with the n-state partition functions.

Let $\pi : \mathcal{M}_n \to \mathcal{M}_n/O_n$ be the corresponding projection function. Then by Theorem 3, any fiber of π intersects $\mathcal{W}_n$. Each fiber of π contains a unique (Zariski-)closed O_n-orbit (cf. [8], [1]). So the Zariski-closed O_n-orbits of $\mathcal{M}_n$ are in one-to-one correspondence with the n-state partition functions. models.

We will show that in fact this orbit is equal to the set of elements in the fiber that belong to $\mathcal{W}_n$. Equivalently, for each n-state partition function f the element R of $\mathcal{W}_n$ with

$f(G) = f_R(G)$ for each virtual link diagram G is unique up to the action of the orthogonal group on R. So $\mathcal{W}_n$ is equal to the union of the closed O_n-orbits.

Theorem 4. *Each $R \in \mathcal{W}_n$ belongs to the unique closed O_n-orbit B contained in $\{R' \mid f_{R'}(G) = f_R(G)$ for each virtual link diagram $G\}$.*

Proof. Suppose not. Then there is a polynomial $q \in \mathcal{O}(\mathcal{M}_n)$ with $q(B) = 0$ and $q(R) \neq 0$. Let U be the O_n-module spanned by $O_n \cdot q$. The morphism $\phi : \mathcal{M}_n \to U^*$ with $\phi(R')(u) = u(R')$ (for $R' \in \mathcal{M}_n$ and $u \in U$) is O_n-equivariant.

Let $\iota : U^* \to T(V)$ be an embedding of U^* as O_n-submodule of $T(V)$. So $\iota \circ \phi$ belongs to $(\mathcal{O}(\mathcal{M}_n) \otimes T(V))^{O_n} = p_n(\mathbb{C}\mathcal{T})$, say $\iota \circ \phi = p_n(\tau)$ with $\tau \in \mathbb{C}\mathcal{T}$. As $\phi(R) \neq 0$ (since $\phi(R)(q) = q(R) \neq 0$), we have $p_n(\tau)(R) \neq 0$. As $R \in \mathcal{W}_n$, there is a tangle T with $p_n(\tau \cdot T)(R) \neq 0$. So $f_R(\tau \cdot T) \neq 0$. However, for any $R' \in B$, $p_n(\tau \cdot T)(R') = p_n(\tau)(R') \cdot p_n(T)(R') = 0$, since $p_n(\tau)(R') = \iota \circ \phi(R') = 0$, as $q(B) = 0$. So $f_{R'}(\tau \cdot T) = 0$ while $f_R(\tau \cdot T) \neq 0$, contradicting the fact that $f_{R'}(G) = f_R(G)$ for each virtual link diagram G. ∎

Corollary 4a. *Let $R, R' \in \mathcal{W}_n$ be such that $f_R(G) = f_{R'}(G)$ for each virtual link diagram G. Then $R' = R^U$ for some $U \in O_n$.*

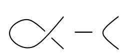
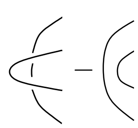
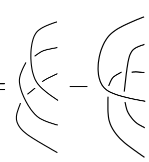
Proof. By Theorem 4, R and R' belong to the same O_n-orbit. ∎

9. Reidemeister moves

Let $\mathcal{V}_n$ be the variety of solutions of (7):

(29) $$\mathcal{V}_n := \{R \mid R \in \mathcal{M}_n,\ R \text{ satisfies (7)}\}.$$

This can be translated into the following quantum tangles, corresponding to the three Reidemeister moves:

(30) $M_1 :=$  , $M_2 :=$  , $M_3 :=$  .

Then (cf. (7)), up to permuting tensor factors, $f_R(M_1) = C(R) - I_n$, $f_R(M_2) = RD(R) - I_n \otimes I_n$, and $f_R(M_3) = E_{1,2}(R)E_{1,3}(R)E_{2,3}(R) - E_{2,3}(R)E_{1,3}(R)E_{1,2}(R)$.

Theorem 5. *Let $f : \mathcal{G} \to \mathbb{C}$ and $n \in \mathbb{N}$. Then $f = f_R$ for some $R \in \mathcal{V}_n$ if and only if f is an n-state partition function and is invariant under Reidemeister moves.*

Proof. Necessity is direct, by definition of $\mathcal{V}_n$. As for sufficiency, by Theorem 3 we can assume that $R \in \mathcal{W}_n$. Since f_R is invariant under Reidemeister moves, we have $f_R(M_i) = 0$ for $i = 1, 2, 3$. Hence $R \in \mathcal{V}_n$. ∎

Let again $\pi : \mathcal{M}_n \to \mathcal{M}_n/O_n$ be the projection function. Then by Theorem 5, any n-state partition function f is a virtual link invariant if and only if the fiber $\pi^{-1}(f)$ intersects $\mathcal{V}_n$.

Now $\mathcal{V}_n$ is an O_n-invariant variety. Then the associated affine variety $\mathcal{V}_n/O_n$ is in one-to-one correspondence with the set of n-state partition functions that are invariant under Reidemeister moves.

Each fiber $\pi^{-1}(f)$ of π contains a unique (Zariski-)closed O_n-orbit. Then f is a virtual link invariant if and only if this closed orbit is a subset of $\mathcal{V}_n$. So there is a one-to-one correspondence between the closed O_n-orbits in $\mathcal{V}_n$ and the Reidemeister move invariant n-state partition functions. Let $\mathcal{V}'_n$ be the set of elements $R \in \mathcal{V}_n$ for which the orbit $O_n \cdot R$ is closed. So $\mathcal{V}'_n = \mathcal{V}_n \cap \mathcal{W}_n$. Then for any $R, R' \in \mathcal{V}'_n$ one has: $f_R(G) = f_{R'}(G)$ for each virtual link diagram if and only if $R' = R^U$ for some $U \in O_n$.

10. The real case

We next give a characterization of virtual link diagram invariants f for which there exists a real $R \in \mathcal{M}_n$ such that $f = f_R$. As we will see, any real R automatically belongs to $\mathcal{W}_n$.

Theorem 6. *Let f be a real-valued virtual link diagram invariant and let $n \in \mathbb{N}$. Then there is a real $R \in \mathcal{M}_n$ with $f = f_R$ if and only if f is multiplicative, $f(\det_{n+1} \cdot T) = 0$ for each $2(n+1)$-tangle T, and the matrix*

$$(f(T \cdot T'))_{T,T' \text{ 4-tangles}} \tag{31}$$

is positive semidefinite.

Proof. Necessity follows from the fact that

$$f_R(T \cdot T') = f_R(T) \cdot f_R(T'). \tag{32}$$

We prove sufficiency. As f is an n-state partition function, there is an algebra homomorphism $\hat{f} : \mathcal{O}(\mathcal{M}_n)^{O_n} \to \mathbb{C}$ such that $\hat{f} \circ p_n = f$. We must show the existence of a real-valued $R \in \mathcal{M}_n$ such that $f_R(G) = f(G)$ for each G; i.e., $p_n(G)(R) = \hat{f} \circ p_n(G)$ for each G; in other words, by Theorem 1: $q(R) = \hat{f}(q)$ for each $q \in \mathcal{O}(\mathcal{M}_n)^{|O_n}$,

By the Procesi-Schwarz theorem [9], such an R exists if and only if

$$\hat{f}(dq \cdot dq) \geq 0 \tag{33}$$

for each $q \in \mathcal{O}(\mathcal{M}_n)^{O_n}$ which is real-valued on real-valued elements of $\mathcal{M}_n$. Here d is the derivative function $\mathcal{O}(\mathcal{M}_n) \to \mathcal{O}(\mathcal{M}_n) \otimes \mathcal{M}_n^*$. The bilinear form $\cdot : (\mathcal{O}(\mathcal{M}_n) \otimes \mathcal{M}_n^*)^2 \to \mathcal{O}(\mathcal{M}_n)$ is defined by $(a \otimes b) \cdot (c \otimes d) := (b \cdot d)ac$ for $a, c \in \mathcal{O}(\mathcal{M}_n)$ and $b, d \in \mathcal{M}_n^*$, where $b \cdot d$ is the standard inner product on $\mathcal{M}_n^*$ (induced by the standard inner product on $\mathbb{C}^n$).

So it suffices to prove (33). We can write

(34) $$q = \sum_G \lambda_G p_n(G),$$

with $\lambda_G \in \mathbb{R}$, and only finitely many of them nonzero. Here G ranges over all virtual link diagrams. Indeed, we can write $q = \sum_G \lambda_G p_n(G)$ with $\lambda_G \in \mathbb{C}$. Then $q = \sum_G \frac{1}{2}(\lambda_G + \overline{\lambda}_G) p_n(G)$ on the real elements of $\mathcal{M}_n$, hence on all elements of $\mathcal{M}_n$.

For any virtual link diagram G, we define a quantum 4-tangle dG. We first define for each $v \in V(G)$, a quantum 4-tangle G_v. Let e_1, e_2, e_3, e_4 be the edges incident with v, in cyclic order, where e_1, e_3 is the over-going pair. Let H be the graph obtained by deleting vertex v and connecting $e_1, \ldots, e_4$ to new vertices $v_1, \ldots, v_4$. Then G_v is defined to be half of the sum of the 4-tangle obtained from H by giving $v_1, \ldots, v_4$ labels $1, \ldots, 4$ respectively, and the 4-tangle obtained from H by giving $v_1, \ldots, v_4$ labels $3, 4, 1, 2$ respectively. Then

(35) $$dG := \sum_{v \in V(G)} G_v.$$

Then for any G, $dp_n(G) = p_n(dG)$, hence for any G, H:

(36) $$dp_n(G) \cdot dp_n(H) = p_n(dG) \cdot p_n(dH) = p_n(dG \cdot dH).$$

So

(37) $$dq \cdot dq = \sum_{G,H} \lambda_G \lambda_H dp_n(G) \cdot dp_n(H) = \sum_{G,H} \lambda_G \lambda_H p_n(dG \cdot dH).$$

Therefore,

(38) $$\hat{f}(dq \cdot dq) = \sum_{G,H} \lambda_G \lambda_H f(dG \cdot dH) \geq 0,$$

by the positive semidefiniteness of (31). This proves '(33). ∎

The positive semidefiniteness condition may be considered as a form of 'reflection positivity'. The proof of the theorem is inspired by the theorem of Szegedy [12] on edge model graph invariants. A similar theorem (but different proof) of Freedman, Lovász, and Schrijver [2] may yield a spin model analogue of Theorem 6.

For real R, (7) is automatically satisfied:

(39) If $R \in \mathcal{M}_n$ is real, then $R \in \mathcal{W}_n$.

Indeed, suppose $f_R(\tau) \neq 0$. Then $f_R(\tau) \cdot f_R(\tau) \neq 0$. Hence $f_R(\tau \cdot \tau) \neq 0$.

This implies:

(40) Let $R \in \mathcal{M}_n$ be real. Then f_R is a virtual link invariant if and only if R belongs to $\mathcal{V}_n$.

Here it suffices to show necessity. If f_R is a virtual link invariant, then, as R belongs to $\mathcal{W}_n$ by (39) and as f_R is invariant under the Reidemeister moves, R satisfies (7). So $R \in \mathcal{V}_n$.

It is an open problem whether for any two nonisotopic virtual link diagrams G and H one has $f_R(G) \neq f_R(H)$ for some $n \in \mathbb{N}$ and some $R \in \mathcal{M}_n$.

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